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Journal of Sound and Vibration 291 (2006) 1239–1254
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Short Communication

Lekhnitskii's classic formula serving as an exact mode shape of simply supported polar orthotropic inhomogeneous circular plates

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Received 18 May 2005; received in revised form 18 May 2005; accepted 26 June 2005
Available online 22 September 2005

Abstract

The semi-inverse problem of free vibration of simply supported inhomogeneous polar orthotropic plates is studied. The mode shape is postulated in the form of the classic formula by Lekhnitskii, namely, the static deflection of the associated uniform polar orthotropic circular plate under uniform loading. The ratios of the circumferential rigidity to the radial rigidity are identified as integer numbers, so that the candidate mode shapes constitute the polynomial functions. The flexural rigidities themselves are also represented by polynomials. The semi-inverse problem of identifying the coefficients of the flexural rigidities is solved analytically. It appears remarkable, that for numerous cases the simple method developed in this study provides novel closed form solutions for the design of the polar orthotropic circular plates with pre-specified mode shapes.

1. Introduction

Since the study on vibrations of circular plates by Chlandi [1] and Poisson [2], numerous studies have been conducted on this subject. The definitive reference in plate vibrations is the monograph by Leissa [3]. Closed-form solution for simply supported isotropic circular plates was reported, apparently for the first time, by Elishakoff and Storch [4].

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2005.06.039

Prior to Ref. [4], various approximate methods have been utilized. Pertinent references for our purpose are those of Jones [5] and Johns [6]. As Leissa [7] mentions, "Jones [5] devised a simple approximate formula for calculating fundamental (i.e., lowest) frequencies if the static deflected shape of a uniformly loaded plate is known. Johns [6] used a two-term solution for static deflection shapes to evaluate frequencies of clamped and simply supported circular plates and compares them with those of Jones [5]". Another study that relates the vibration frequency of plates and their deflection was presented by Mazumdar [8].

An unconventional application of static deflections of circular plates to the vibration problems was conducted in a recent study [4] (see also a recent monograph [9]). In the papers several semi-inverse problems were posed and solved. Namely, it was postulated that the expression in the square parentheses for displacement of the *homogeneous* and *uniform* circular plate under loading q

$$w = \left(\frac{q}{64D}\right)\left[\left(R^2 - r^2\right)\left(\frac{5+v}{1+v}R^2 - r^2\right)\right] \tag{1}$$

can serve as a mode shape of an *inhomogeneous* isotropic plate.

In this paper, Ref. [9] is generated to the simply supported polar-orthotropic circular plate. Several closed-form solutions are found, apparently for the first time in the literature.

2. Basic equations

The governing differential equation of the polar orthotropic circular plate of varying flexural rigidity reads

$$\begin{aligned} &rD_r W^{IV} + \left(2D_r + 2r\frac{\mathrm{d}D_r}{\mathrm{d}r} + v_\theta D_r - v_r D_\theta\right) W''' \\ &- \left[\frac{\mathrm{d}}{\mathrm{d}r}\left(-D_r - r\frac{\mathrm{d}D_r}{\mathrm{d}r} - v_\theta D_r + v_r D_\theta\right) - v_\theta \frac{\mathrm{d}D_r}{\mathrm{d}r} + \frac{1}{r}D_\theta\right] W'' \\ &- \left[\frac{\mathrm{d}}{\mathrm{d}r}\left(-v_\theta \frac{\mathrm{d}D_r}{\mathrm{d}r} + \frac{1}{r}D_\theta\right)\right] W' = r\rho h \omega^2 W(r), \end{aligned} \tag{2}$$

where $W(r)$ is the mode shape, D_r and D_θ are, respectively, radial and circumferential flexural rigidities, v_r, v_θ the Poisson's ratios of orthotropic plate, r the radial coordinate, ρ the material density, h the thickness, and ω the natural frequency.

Our analysis is heavily based on the application of the classic formula by Leckhnitskii [10]

$$w(r) = \frac{q_0 a^4}{8(9-k^2)D_r}\left[\frac{(3-k)(4+k+v_\theta)}{(1+k)(k+v_\theta)} - \frac{4(3+v_\theta)}{(1+k)(k+v_\theta)}\left(\frac{r}{R}\right)^{k+1} + \left(\frac{r}{R}\right)^4\right] \tag{3}$$

for the deflection of the circular simply supported polar orthotropic plate under uniformly distributed load q_0. In Eq. (3) $k^2 = D_\theta / D_r$ is the rigidity ratio. We would like to design a circular plate such that is fundamental mode shape, $W(r)$ in Eq. (2) would coincide with the expression in brackets in Eq. (3), namely,

$$W(r) = \frac{(3-k)(4+k+v_\theta)}{(1+k)(k+v_\theta)} - \frac{4(3+v_\theta)}{(1+k)(k+v_\theta)}\left(\frac{r}{R}\right)^{k+1} + \left(\frac{r}{R}\right)^4. \quad (4)$$

As the literature shows, the postulate in Eq. (3) cannot be realized for the homogeneous plates, with $D_r(r) = const.$, $D_\theta(r) = const.$ and $\rho(r) = const.$ However, if previous experience for isotropic case is any guide [9–11], we can anticipate finding *heterogeneous* plates that possess expression in Eq. (3) as the mode shape.

We specify the value k in Eq. (2) as equal to some integer m and investigate if there exists a non-negative-valued distributions of $D_r(r)$ and $D_\theta(r)$ that taken together with mode shape as follows:

$$W(r) = \frac{1}{(9-m^2)}\frac{(3-m)(4+m+v_\theta)}{(1+m)(m+v_\theta)} - \frac{4(3+v_\theta)}{(1+m)(m+v_\theta)}\left(\frac{r}{R}\right)^{m+1} + \left(\frac{r}{R}\right)^4 \quad (5)$$

satisfy Eq. (2).

3. Semi-inverse method of solution associated with $m = 1$

For $m = 1$, the mode shape in Eq. (5) reads

$$W(r) = \frac{1}{8}\left[\frac{10+2v_\theta}{2+2v_\theta} - \frac{(12+4v_\theta)}{(2+2v_\theta)}\left(\frac{r}{R}\right)^3 + \left(\frac{r}{R}\right)^4\right]. \quad (6)$$

The flexural rigidities are sought as polynomials of the fourth order

$$D_r(r) = b_0 + b_1 r + b_2 r^2 + b_3 r^3 + b_4 r^4, \quad (7)$$

$$D_\theta(r) = k^2 D_r(r), \quad (8)$$

where k is taken as constant. The substitution of Eqs. (6)–(8) into Eq. (2) leads to the following polynomial equation:

$$A_0 + A_1 r + A_2 r^2 + A_3 r^3 + A_4 r^4 + A_5 r^5 = 0, \quad (9)$$

where

$$\begin{aligned} A_0 = {} & -8v_\theta^2 R^2 b_1 + 12k^2 R^2 b_1 - 32 v_\theta R^2 b_1 + 4 v_\theta v_r R^2 k^2 b_1 + 4k^2 v_\theta R^2 b_1 \\ & + 12k^2 R^2 v_r b_1 - 24R^2 b_1, \end{aligned} \quad (10)$$

$$\begin{aligned} A_1 = {} & 24k^2 v_r R^2 b_2 + 24k^2 R^2 b_2 - 24k^2 v_r v_\theta b_0 + 8k^2 R^2 v_\theta b_2 - 8k^2 b_0 - 72R^2 b_2 \\ & - 24k^2 v_r b_0 + 96 v_\theta b_0 + 24 v_\theta^2 b_0 + 72 b_0 - 24R^2 v_\theta^2 b_2 - 8k^2 v_\theta b_0 \\ & + 8k^2 v_\theta v_r R^2 b_2 - 96R^2 v_\theta b_2 - \rho h\omega^2 R^4 v_\theta - 5\rho h\omega^2 R^4, \end{aligned} \quad (11)$$

$$\begin{aligned} A_2 = {} & -36k^2 v_r v_\theta b_1 + 48 v_\theta^2 b_1 - 144R^2 b_3 + 12k^2 R^2 v_\theta b_3 + 12R^2 k^2 v_r v_\theta b_3 + 192 v_\theta b_1 \\ & - 12k^2 b_1 + 144 b_1 - 192 v_\theta R^2 b_3 - 36k^2 v_r b_1 - 48R^2 v_\theta^2 b_3 - 12k^2 v_\theta b_1 \\ & + 36k^2 R^2 b_3 + 36k^2 R^2 v_r b_3, \end{aligned} \quad (12)$$

$$
\begin{aligned}
A_3 = & -48k^2 v_r v_\theta b_2 - 240R^2 b_4 - 80v_\theta^2 R^2 b_4 + 48k^2 R^2 b_4 - 16k^2 b_2 + 320v_\theta b_2 \\
& - 48v_r k^2 b_2 - 16v_\theta k^2 b_2 + 80v_\theta^2 b_2 - 320R^2 v_\theta b_4 + 240b_2 + 16k^2 R^2 v_r v_\theta b_4 \\
& + 16k^2 R^2 v_\theta b_4 + 48k^2 R^2 v_r b_4 + 2\rho h\omega^2 R^2 v_\theta + 6\rho h\omega^2 R^2,
\end{aligned} \tag{13}
$$

$$
A_4 = 120v_\theta^2 b_3 + 360b_3 - 60k^2 v_r v_\theta b_3 - 60k^2 v_r b_3 - 20k^2 v_\theta b_3 + 480v_\theta b_3 - 20k^2 b_3, \tag{14}
$$

$$
\begin{aligned}
A_5 = & \, 504b_4 - 72k^2 v_r v_\theta b_4 + 672v_\theta b_4 + 168v_\theta^2 b_4 - 24k^2 b_4 - 24k^2 v_\theta b_4 \\
& - \rho h\omega^2 v_\theta - \rho h\omega^2.
\end{aligned} \tag{15}
$$

From Eq. (15) we get the relationship between the natural frequency squared ω^2 and the coefficient b_4

$$
\omega^2 = 24(21 + 7v_\theta - 3v_r k^2 - 21k^2)b_4/\rho h. \tag{16}
$$

Eq. (14) shows that b_3 vanishes identically. The equation resulting from substitution of Eq. (16) into Eq. (13), results in the formula for b_2, as related to b_4

$$
b_2 = -2\frac{(3k^2 + 12k^2 v_r + 4k^2 v_r v_\theta - 8v_\theta^2 + k^2 v_\theta - 53v_\theta - 87)}{(1 + v_\theta)(-5v_\theta + k^2 + 3k^2 v_r - 15)} R^2 b_4. \tag{17}
$$

Eq. (12) leads to the conclusion that $b_1 = 0$. The equation resulting for substitution of Eq. (16) and (17) into Eq. (11) leads to the expression for b_0

$$
\begin{aligned}
b_0 = & -\frac{b_4}{(1 + v_\theta)(k^2 - 5v_\theta + 3k^2 v_r - 15)(k^2 - 3v_\theta + 3k^2 v_r - 9)} \\
& \times (-882k^2 v_r - 1632k^2 v_r v_\theta - 246k^2 v_\theta + 5778v_\theta + 36k^2 - 650v_\theta^2 k^2 v_r + 3312v_\theta^2 \\
& - 68v_\theta^3 k^2 v_r + 48k^4 v_\theta v_r + 144k^4 v_r^2 v_\theta + 19k^4 v_r^2 v_\theta^2 + 8k^4 v_r v_\theta^2 - 128v_\theta^2 k^2 \\
& - 12v_\theta^3 k^2 + 6k^4 v_\theta + 63k^4 v_r^2 + k^4 v_\theta^2 + 750v_\theta^3 + 57v_\theta^4 - 3k^4 + 3159).
\end{aligned} \tag{18}
$$

In the view of the relationship

$$
v_\theta = k^2 v_r, \tag{19}
$$

we obtain the following final expression for the flexural rigidity:

$$
\begin{aligned}
D_r(r) = & \frac{b_4}{(1 + k^2 v_r)^2 (2k^2 v_r - k^2 + 15)(k^2 - 9)} \times [(-135 + 50k^4 v_r + 28k^6 v_r^2 - k^4 + 24k^2 \\
& - 171k^4 v_r^2 - 288k^2 v_r - 18k^6 v_r^3 + 2k^8 v_r^3 - k^8 v_r^2 - 2k^6 v_r) + (6k^4 - 108k^6 v_r^2 \\
& + 2304k^8 v_r^2 + 72k^6 v_r^3 - 8k^8 v_r^3 + 1566 + 8k^6 v_r + 810k^4 v_r^2 - 328k^4 v_r - 228k^2)R^2 r^2 \\
& + (-k^8 v_r^2 - 4896k^2 v_r - 1734k^4 v_r^2 - 3159 + 246k^4 v_r - 8k^8 v_r^4 - 214k^6 v_r^3 + 80k^6 v_r^2 \\
& + 2k^4 - 36k^2 + 6k^8 v_r^3 - 6k^6 v_r)R^4 r^4].
\end{aligned} \tag{20}
$$

Fig. 1 depicts the variation of $D_r(r)$ as a function of r, for various values of k, for v_r fixed at 0.35.

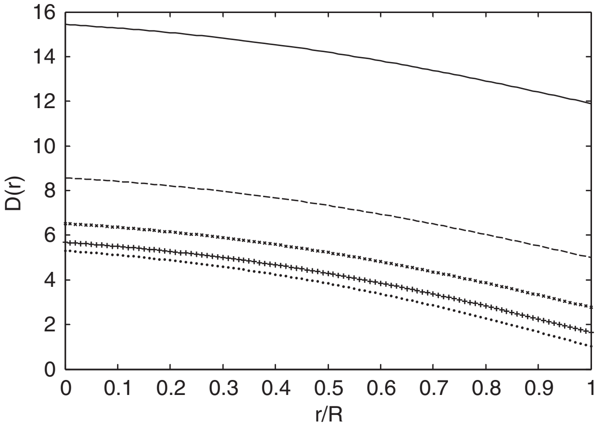

Fig. 1. Variation of $D(r)$ versus nondimensional radial coordinate r/R for various values of k, and $v_r = 0.35$.

4. Semi-inverse method of solution associated with $m = 2$

For $m = 2$, the mode shape in Eq. (5) reads

$$W(r) = \frac{1}{5}\left[\frac{6 + v_\theta}{6 + 3v_\theta} - \frac{(12 + 4v_\theta)}{(6 + 3v_\theta)}\left(\frac{r}{R}\right)^3 + \left(\frac{r}{R}\right)^4\right]. \tag{21}$$

The result of substitution of Eq. (21) in coincidence with Eqs. (7) and (8) yields

$$B_0 + B_1 r + B_2 r^2 + B_3 r^3 + B_4 r^4 + B_5 r^5 = 0, \tag{22}$$

where

$$\begin{aligned} B_0 = &-24v_\theta^2 Rb_0 + 36k^2 Rb_0 - 144Rb_0 + 24v_r v_\theta Rk^2 b_1 + 72k^2 v_r Rb_0 \\ &+ 12k^2 Rv_\theta b_0 - 120Rv_\theta b_0, \end{aligned} \tag{23}$$

$$\begin{aligned} B_1 = &-48k^2 b_0 - 144k^2 v_r b_0 + 72v_\theta^2 b_0 - 432Rb_1 + 432b_0 - 72k^2 v_r v_\theta b_0 \\ &+ 72k^2 Rb_1 - 72R^2 v_\theta^2 b_1 + 48k^2 v_r v_\theta Rb_1 - 360Rv_\theta b_1 + 360v_\theta b_0 \\ &- 24k^2 v_\theta b_0 + 144k^2 v_r Rb_1 + 24k^2 v_\theta R - \rho h\omega^2 R^4 v_\theta - 6\rho h\omega^2 R^4, \end{aligned} \tag{24}$$

$$\begin{aligned} B_2 = &-720Rv_\theta b_2 + 108k^2 Rb_2 - 108v_r v_\theta k^2 b_1 - 72k^2 b_1 + 36Rk^2 v_\theta b_2 + 864b_1 \\ &- 144v_\theta^2 Rb_2 + 720v_\theta b_1 + 72k^2 v_r v_\theta Rb_2 + 216k^2 v_r Rb_2 - 36k^2 v_\theta b_1 \\ &- 864Rb_2 + 144v_\theta^2 b_1 - 216k^2 v_2 b_1, \end{aligned} \tag{25}$$

$$\begin{aligned} B_3 = &\, 48k^2 Rv_\theta b_3 - 1200v_\theta Rb_3 + 96v_r v_\theta Rk^2 b_3 + 144k^2 Rb_3 + 1440b_2 - 288k^2 v_r b_2 \\ &- 144v_r v_\theta k^2 b_2 + 288v_r k^2 Rb_3 - 48k^2 v_\theta b_2 - 96k^2 b_2 + 1200v_\theta b_2 - 1440Rb_3 \\ &+ 240v_\theta^2 b_2 - 240Rv_\theta^2 b_3, \end{aligned} \tag{26}$$

$$B_4 = -60k^2 v_\theta b_3 - 120k^2 b_3 + 180Rk^2 b_4 - 216Rb_4 + 2160b_3 + 60k^2 R v_\theta b_4 - 180k^2 v_r v_\theta b_3 - 360R v_\theta^2 b_4 + 120k^2 R v_r v_\theta b_4 + 360k^2 R v_r b_4 - 1800R v_\theta b_4 + 1800 v_\theta b_3 + 360 v_\theta^2 b_3 - 360k^2 v_r b_3 + 12\rho h \omega^2 R + 4\rho h \omega^2 R v_\theta, \tag{27}$$

$$B_5 = 3024b_4 + 504v_\theta^2 b_4 - 72v_\theta k^2 b_4 - 144k^2 b_4 - 216k^2 v_r v_\theta b_4 - 432k^2 v_r b_4 + 2520 v_\theta b_4 - 3\rho h \omega^2 v_\theta - 6\rho h \omega^2. \tag{28}$$

From Eq. (28) we get the expression for natural frequency squared ω^2

$$\omega^2 = 24(21 + 7v_\theta - 3v_r k^2 - 21k^2)b_4/\rho h. \tag{29}$$

It is remarkable, that the expression (29) coincides with Eq. (16) for the squared natural frequency, although it is derived for another mode shape. Substitution of Eq. (29) into Eq. (27) leads to

$$b_3 = -\frac{1}{5}\frac{(9k^2 + 42k^2 v_r + 14k^2 v_r v_\theta - 26v_\theta^2 + 3k^2 v_\theta - 186v_\theta - 324)}{(2 + v_\theta)(-6v_\theta + k^2 + 3k^2 v_r - 18)} R b_4. \tag{30}$$

Substituting Eq. (30) into Eq. (26) we obtain

$$b_2 = -\frac{1}{5(2+v_\theta)(k^2 - 6v_\theta + 3k^2 v_r - 18)(k^2 - 5v_\theta + 3k^2 v_r - 15)} \times (-186v_\theta + 9k^2 - 324 + 42k^2 v_r + 14k^2 v_r v_\theta - 26v_\theta^2 + 3k^2 v_\theta) \times (-25v_\theta + k^2 v_\theta + 6k^2 v_r + 2k^2 v_r v_\theta - 5v_\theta^2 + 3k^2 - 30). \tag{31}$$

Eq. (25) leads to the coefficient b_1

$$b_1 = -\frac{b_4 R^2}{5(2+v_\theta)^2(-6v_\theta + k^2 + 3k^2 v_r - 18)(-5v_\theta + k^2 + 3k^2 v_r - 15)} \times (-186v_\theta + 9k^2 - 324 + 14k^2 v_r v_\theta - 26v_\theta^2 + 3k^2 v_\theta)(-30 + 3k^2 + k^2 v_\theta + 6k^2 v_r + 2k^2 v_r v_\theta - 25v_\theta - v_\theta^2). \tag{32}$$

Note that from Eq. (23), we conclude that $b_0 = 0$. Eq. (24) yields another expression for b_1

$$b_1 = -\frac{b_4(-7v_\theta + k^2 + 3k^2 v_r - 21)(6 + v_\theta)R^3}{-18 + k^2 v_\theta + 2k^2 v_r v_\theta + 6k^2 v_r + 3k^2 - 15v_\theta - 3v_\theta^2}. \tag{33}$$

If the expression in Eqs. (32) and (33) differ from each other, the problem contains a contradiction. In order to have a consistent set, we demand the expression for b_1, in Eq. (32) and in Eq. (33) to be equal. This results in the following polynomial equation:

$$\sum_{j=0}^{8} C_j v_r^j = 0, \tag{34}$$

where

$$C_0 = -3k^8 - 1260k^6 + 131\,004k^4 - 2\,372\,976k^2 + 12\,130\,560, \tag{35}$$

$$C_1 = 76k^{10} - 6930k^8 + 320\,580k^6 - 5\,285\,304k^4 + 27\,511\,488k^2, \tag{36}$$

$$C_2 = 78k^{12} - 6880k^{10} + 277\,215k^8 - 4\,609\,980k^6 + 25\,441\,776k^4, \tag{37}$$

$$C_3 = 24k^{14} - 2580k^{12} + 111\,200k^{10} - 2\,031\,330k^8 + 12\,453\,480k^6, \tag{38}$$

$$C_4 = 2k^{16} - 390k^{14} + 21\,885k^{12} - 485\,570k^{10} + 3\,511\,368k^8, \tag{39}$$

$$C_5 = -20k^{16} + 2028k^{14} - 62\,892k^{12} + 582\,054k^{10}, \tag{40}$$

$$C_6 = 70k^{16} - 4068k^{14} - 54\,993k^{12}, \tag{41}$$

$$C_7 = -100k^{16} + 2658k^{14}, \tag{42}$$

$$C_8 = 48k^6. \tag{43}$$

Numerical evaluation of the roots of Eq. (34) reveals the following: for $k = 1$, 2 or 3 it does not possess real positive roots for ν_r. For $k = 4$ the only positive root of Eq. (34) is $\nu_r = 0.11086$. Fig. 2 depicts the appropriate variation of the radial flexural rigidity $D_r(r)$. The values of the circumferential rigidity are obtainable by dividing $D_r(r)$ by the appropriate value of k^2. For $k = 5$, Eq. (34) yields $\nu_r = 0.44134$ (Fig. 3); $k = 6$ corresponds to $\nu_r = 0.561495$ (Fig. 4); Eq. (34) has two positive roots $\nu_r = 0.295185$ 0.6490527 for $k = 7$. Two resulting curves for $D_r(r)$ are shown in Fig. 5. Likewise, two roots $\nu_r = 0.03256135$ and 0.71579117 correspond to $k = 8$ (Fig. 6). The value $k = 9$ is associated with roots $\nu_r = 0.3507312$ and 0.7670678 (Fig. 7), whereas $k = 10$ corresponds to $\nu_r = 0.37285365$ and 0.80664837 (Fig. 8).

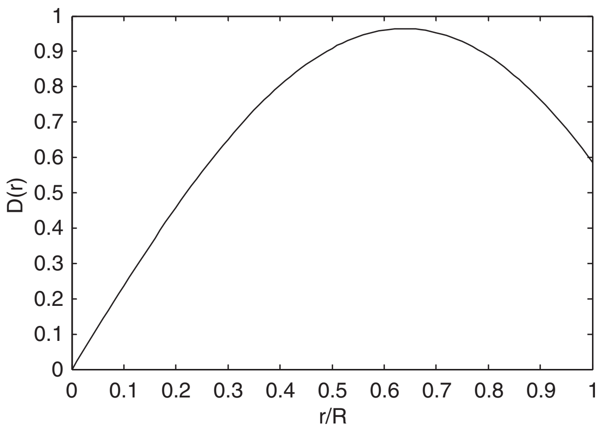

Fig. 2. Variation of $D(r)$ versus nondimensional radial coordinate r/R for $k = 4$ and $\nu_r = 110866$.

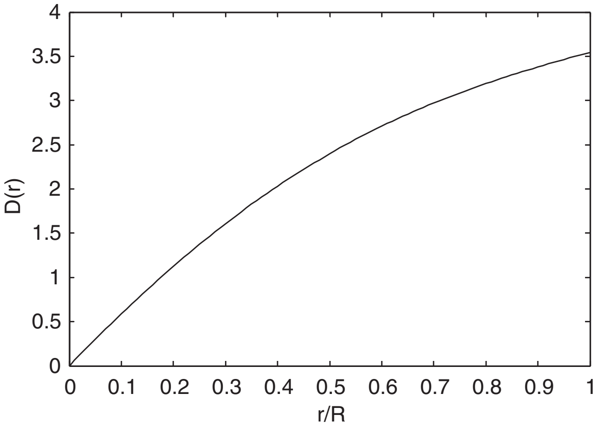

Fig. 3. Variation of $D(r)$ versus nondimensional radial coordinate r/R for $k = 5$ and $v_r = 0.44134$.

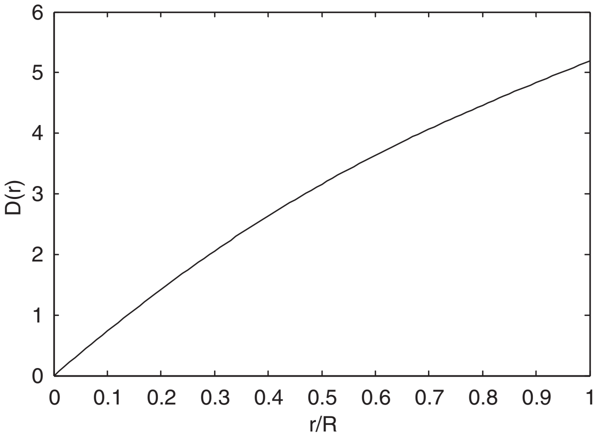

Fig. 4. Variation of $D(r)$ versus nondimensional radial coordinate r/R for $k = 6$ and $v_r = 0.561495$.

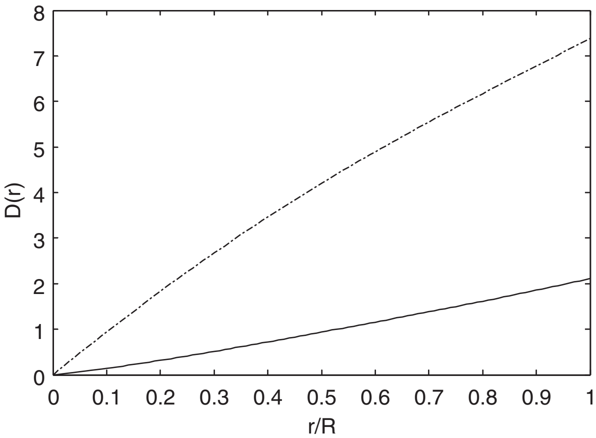

Fig. 5. Variation of $D(r)$ versus nondimensional radial coordinate r/R for $k = 7$ (-$v_r = 0.295815$, -.-$v_r = 0.3507312$).

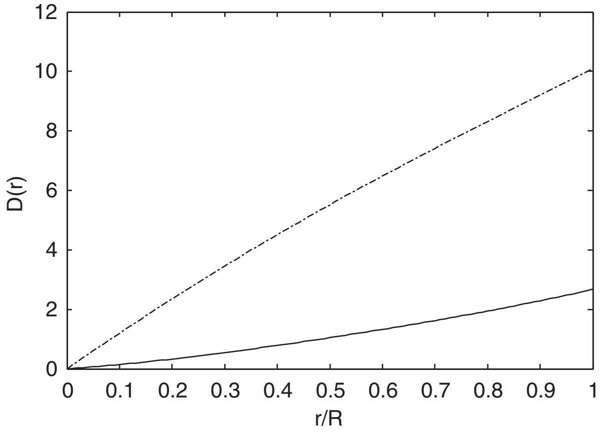

Fig. 6. Variation of $D(r)$ versus nondimensional radial coordinate r/R for $k = 8$ (-$v_r = 0.3256135$, -.-$v_r = 0.71579117$).

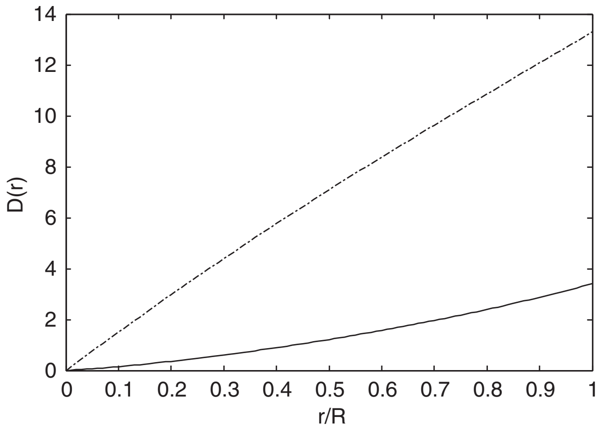

Fig. 7. Variation of $D(r)$ versus nondimensional radial coordinate r/R for $k = 9$ (-$v_r = 0.3507312$, -.-$v_r = 0.7670678$).

5. Semi-inverse method of solution associated with $m = 3$

As is seen from Eq. (3) when m tends to three, the denominator in front of the square parenthesis approaches zero. The first term in the square parentheses tends to zero, but the product

$$\lim_{m\to 3} \frac{1}{(9-m^2)} \frac{(3-m)(4+m+v_\theta)}{(1+m)(m+v_\theta)} = \frac{1}{54+10v_\theta} \tag{44}$$

turns out to approach a finite value, $(54+10v_\theta)^{-1}$.

The sum of the second and the third terms in the square parenthesis in Eq. (5) also tends to zero. As Lekhnitskii [10] stresses, "formulas for the case when $m = 3$ will be found by going to the

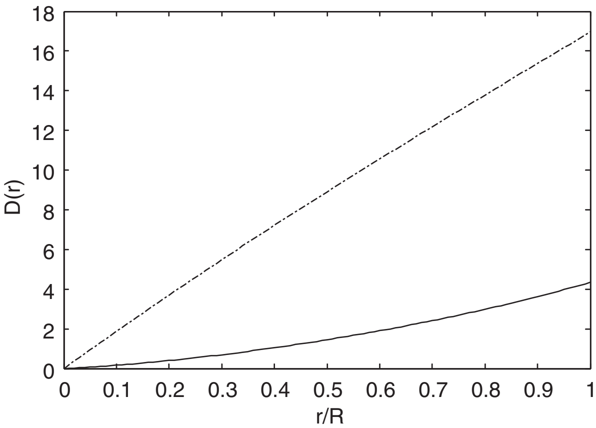

Fig. 8. Variation of $D(r)$ versus nondimensional radial coordinate r/R for $k = 10$ (-$v_r = 0.37285365$, -.-$v_r = 0.80664837$).

limit''. Indeed,

$$\lim_{m\to 3} \frac{1}{(9-m^2)}\left[\left(\frac{r}{R}\right)^4 - \frac{4(3+v_\theta)}{(1+m)(m+v_\theta)}\left(\frac{r}{R}\right)^{m+1}\right] = \frac{r^4[-7 - v_\theta - 12\log(R) - 4\log(R)v_\theta + 12\log(r) + 4\log(r)v_\theta]}{24R^4(3+v_\theta)} \tag{45}$$

using the L'Hopital's rule.

Then, the candidate mode shape for $m = 3$ becomes

$$W(r) = \frac{1}{(54+10v_\theta)} + \frac{r^4\left[-7 - v_\theta - (12+4v_\theta)\ln(r/R)\right]}{24R^4(3+v_\theta)}. \tag{46}$$

Note that Ari-Gur and Stavsky [11] also report displacements including logarithmic function of the radial coordinate for $k = 3$. The mode shape (46) does not constitute a polynomial function; since in this study we limit ourselves with polynomial mode shapes, the reconstruction of flexural rigidities for the case $m = 3$ will not be pursued. This does not mean that there are no nonpolynomial mode shapes for heterogeneous structures; for nonpolynomial mode shapes in the semi-inverse problems one can consult Elishakoff's [9] monograph, as well as the paper by Caliò and Elishakoff [12]. In the former [9] the rational functions are obtained as mode shapes, whereas in the latter [12] trigonometric mode shapes are utilized.

6. Semi-inverse method of solution associated with $m = 4$

For $m = 4$, the mode shape in Eq. (5) becomes

$$W(r) = \frac{(-8 - v_\theta)}{(20+5v_\theta)} + \left(\frac{r}{R}\right)^4 - \frac{(12+4v_\theta)}{(20+5v_\theta)}\left(\frac{r}{R}\right)^5. \tag{47}$$

The result of substitution of Eq. (21) in coincidence of Eqs. (7) and (8) yields

$$E_0 + E_1 r + E_2 r^2 + E_3 r^3 + E_4 r^4 + E_5 r^5 + E_6 r^6 = 0, \tag{48}$$

where

$$\begin{aligned} E_0 = {} & 8\rho h\omega^2 R^5 + 1440 b_0 R + \rho h\omega^2 R^5 v_\theta + 840 b_0 R v_\theta + 120 b_0 v_\theta^2 R - 40k^2 b_0 R v_\theta \\ & - 160k^2 R b_0 - 480k^2 b_0 R v_r - 120 b_0 R k^2 v_r v_\theta, \end{aligned} \tag{49}$$

$$\begin{aligned} E_1 = {} & 2880 b_1 R + 60 b_0 k^2 v_\theta + 1680 R b_1 v_\theta - 240 b_0 v_\theta^2 - 720k^2 b_1 v_r R - 1680 b_0 v_\theta \\ & - 60k^2 R v_\theta b_1 + 180k^2 b_0 + 240 b_0 k^2 v_r v_\theta - 240k^2 b_1 R - 180 b_1 k^2 v_r v_\theta R \\ & + 240 b_1 R v_\theta^2 - 2880 b_0 + 720k^2 b_0 v_r, \end{aligned} \tag{50}$$

$$\begin{aligned} E_2 = {} & 960k^2 b_1 v_r - 2800 b_1 v_\theta + 80k^2 b_1 v_\theta + 240k^2 b_1 - 4800 b_1 - 960k^2 b_2 v_r R \\ & + 2800 R v_\theta b_2 - 240k^2 b_2 R v_r v_\theta + 400 b_2 R v_\theta^2 - 80k^2 b_2 R v_\theta + 4800 b_2 R \\ & - 400 b_1 v_\theta^2 - 320k^2 b_2 R + 320k^2 b_1 v_r v_\theta, \end{aligned} \tag{51}$$

$$\begin{aligned} E_3 = {} & 7200 b_3 R + 4200 b_3 R v_\theta + 400k^2 b_2 v_r v_\theta - 4200 b_2 v_\theta - 400k^2 b_3 R - 600 b_2 v_\theta^2 \\ & + 300k^2 b_2 - 7200 b_2 - 100k^2 b_3 R v_\theta + 1200k^2 b_2 v_r - 1200k^2 b_3 v_r R \\ & + 600 b_3 R v_\theta^2 - 300k^2 b_3 R v_r v_\theta + 100k^2 b_2 v_\theta, \end{aligned} \tag{52}$$

$$\begin{aligned} E_4 = {} & 10\,080 b_4 R - 20\rho h\omega^2 R - 840 b_3 v_\theta^2 + 840 b_4 R v_\theta^2 - 5880 b_3 v_\theta + 5880 b_4 R v_\theta \\ & - 10\,080 b_3 - 5\rho h\omega^2 R v_\theta + 360k^2 b_3 - 1440k^2 b_4 v_r R + 480k^2 b_3 v_r v_\theta \\ & + 120k^2 b_3 v_\theta - 120k^2 b_4 R v_\theta + 1440k^2 b_3 v_r - 360k^2 b_4 v_r R v_\theta - 480k^2 b_4 R, \end{aligned} \tag{53}$$

$$\begin{aligned} E_5 = {} & 560k^2 b_4 v_r v_\theta + 1680k^2 b_4 v_r - 20 b_4 v_\theta^2 + 140k^2 b_4 v_\theta + 12\rho h\omega^2 - 13\,440 b_4 \\ & + 4\rho h\omega^2 v_\theta + 420k^2 b_4 - 7840 b_4 v_\theta. \end{aligned} \tag{54}$$

From Eq. (54) we get the expression for natural frequency squared ω^2

$$\omega^2 = 35(32 + 8v_\theta - 4v_r k^2 - k^2) b_4/\rho h. \tag{55}$$

Substitution of Eq. (55) into Eq. (53) leads to

$$b_3 = -\frac{1}{24}\frac{(11k^2 + 68k^2 v_r - 112 v_\theta - 616)(4 + v_\theta)}{(-49 v_\theta + 3k^2 + 12k^2 v_r + k^2 v_\theta + 4k^2 v_r v_\theta - 7v_\theta^2 - 84)} R b_4. \tag{56}$$

Substituting Eq. (56) into Eq. (52) we obtain

$$b_1 = F/G, \tag{57}$$

$$\begin{aligned} F = {} & -(-6v_\theta^2 + 12k^2 v_r - 42 v_\theta + 3k^2 v_r v_\theta + 4k^2 + k^2 v_\theta - 72)(-112 v_\theta - 616 + 11k^2 + 68k^2 v_r) \\ & \times (4 + v_\theta)^2 R^2 b_4, \end{aligned} \tag{58}$$

$$G = 24(-7v_\theta^2 + 12k^2v_r - 49v_\theta + 4k^2v_rv_\theta + 3k^2 + k^2v_\theta - 84)(-6v_\theta^2 + 4k^2v_rv_\theta - 42v_\theta + 12k^2v_r + 3k^2 + k^2v_\theta - 72). \tag{59}$$

Eq. (51) leads to the coefficient b_1

$$b_1 = H/I, \tag{60}$$

where

$$H = -(-5v_\theta^2 + 12k^2v_r - 35v_\theta + 3k^2v_rv_\theta + 4k^2 + k^2v_\theta - 60)(-6v_\theta - 18 + k^2 + 3k^2v_r) \times (-112v_\theta - 616 + 11k^2 + 68k^2v_r)(4 + v_\theta)^2R^3b_4, \tag{61}$$

$$I = 24(-7v_\theta^2 + 12k^2v_r - 49v_\theta + 4k^2v_rv_\theta + 3k^2 + k^2v_\theta - 84)(-6v_\theta^2 + 4k^2v_rv_\theta - 42v_\theta + 12k^2v_r + 3k^2 + k^2v_\theta - 72)(3k^2 + 12k^2v_r - 35v_\theta + k^2v_\theta^2 + 4k^2v_rv_\theta - 60). \tag{62}$$

From Eq. (50), we obtain

$$b_0 = K/L, \tag{63}$$

where

$$K = -(-5v_\theta^2 + 12k^2v_r - 35v_\theta + 3k^2v_rv_\theta + 4k^2 + k^2v_\theta - 60)(-6v_\theta - 18 + k^2 + 3k^2v_r) \times (-112v_\theta - 616 + 11k^2 + 68k^2v_r)(-5v_\theta + 3k^2v_r + k^2 - 15)(4 + v_\theta)^3R^4b_4, \tag{64}$$

$$L = 24(-7v_\theta^2 + 12k^2v_r - 49v_\theta + 4k^2v_rv_\theta + 3k^2 + k^2v_\theta - 84)(-6v_\theta^2 + 4k^2v_rv_\theta - 42v_\theta + 12k^2v_r + 3k^2 + k^2v_\theta - 72)(3k^2 + 12k^2v_r - 35v_\theta + k^2v_\theta^2 + 4k^2v_rv_\theta - 60) \times (k^1v_\theta - 4v_\theta^2 + 4k^2v_rv_\theta - 28v_\theta + 3k^2 + 12k^2v_r - 48). \tag{65}$$

Eq. (49) yields another expression for b_0

$$b_0 = -\frac{7(-8v_\theta + k^2 + 4k^2v_r - 32)(8 + v_\theta)R^4b_4}{(-3v_\theta + 3k^2v_r + k^2 - 9)(4 + v_\theta)}. \tag{66}$$

We demand the expression for b_0, in Eqs. (63) and (66) to be equal. This results in the following polynomial equation:

$$\sum_{j=0}^{9} M_jv_r^j = 0, \tag{67}$$

where

$$M_0 = -393\,920k^8 + 31\,167\,360k^6 - 1\,023\,206\,400k^4 + 1\,455\,368\,6016k^2 - 75\,246\,796\,800, \tag{68}$$

$$M_1 = 5765k^{12} - 856\,040k^{10} + 58\,795\,440k^8 - 1\,855\,455\,360k^6 + 26\,477\,328\,384k^4 - 13\,942\,554\,620k^2, \tag{69}$$

$$M_2 = 4300k^{14} - 644\,050k^{12} + 42\,939\,440k^{10} - 1\,365\,596\,320k^8 + 20\,040\,795\,648k^6 - 109\,161\,271\,296k^4, \tag{70}$$

$$M_3 = 1390k^{16} - 227\,400k^{14} + 15\,828\,175k^{12} - 529\,797\,400k^{10} + 8\,218\,352\,640k^8 - 47\,164\,667\,904k^6, \tag{71}$$

$$M_4 = 200k^{18} - 40\,100k^{16} + 3\,160\,400k^{14} - 117\,354\,050k^{12} + 1\,992\,532\,896k^{10} - 12\,332\,498\,688k^8, \tag{72}$$

$$M_5 = 10k^{20} - 3310k^{18} + 337\,685k^{16} - 15\,019\,770k^{14} - 292\,908\,120k^{12} - 2\,015\,921\,280k^{10}, \tag{73}$$

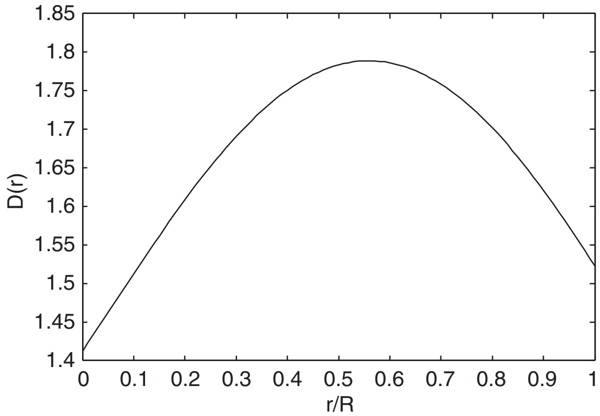

Fig. 9. Variation of $D(r)$ versus nondimensional radial coordinate r/R for $k = 5$ and $v_r = 0.086627$.

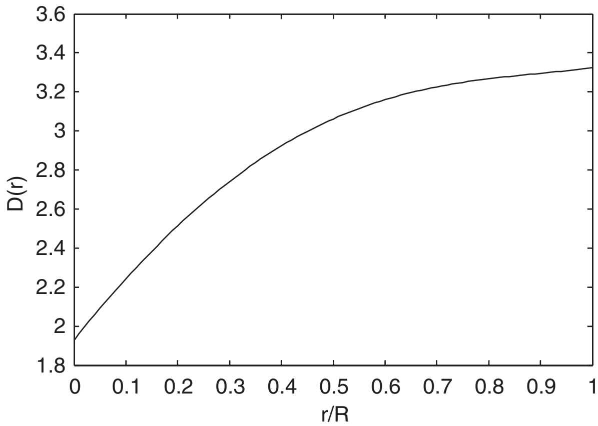

Fig. 10. Variation of $D(r)$ versus nondimensional radial coordinate r/R for $k = 6$ and $v_r = 0.362716$.

$$M_6 = -100k^{20} + 17\,740k^{18} - 1\,072\,240k^{16} + 25\,774\,302k^{14} - 205\,541\,136k^{12}, \tag{74}$$

$$M_7 = 350k^{20} - 38\,200k^{18} + 1\,281\,213k^{16} - 12\,579\,120k^{14}, \tag{75}$$

$$M_8 = -500k^{20} + 31\,086k^{18} - 417\,732k^{16}, \tag{76}$$

$$M_9 = 240k^{20} - 5688k^{18}. \tag{77}$$

Numerical evaluation of the roots of Eq. (67) reveals the following: for $k = 1$, 2 or 3 Eq. (67) does not possess real positive roots for v_r. For $k = 4$ the positive roots of Eq. (67) are $v_r = \frac{1}{32}$ and $\frac{1}{4}$; however, the flexural rigidities associated with either of the value turns out to be negative. Thus, for $k = 4$ no physically realizable solution is found. For $k = 5$, Eq. (67) yields $v_r = 0.086627$ (Fig. 9) and 0.199999, the latter value not yielding the positive-valued flexural rigidity. The value

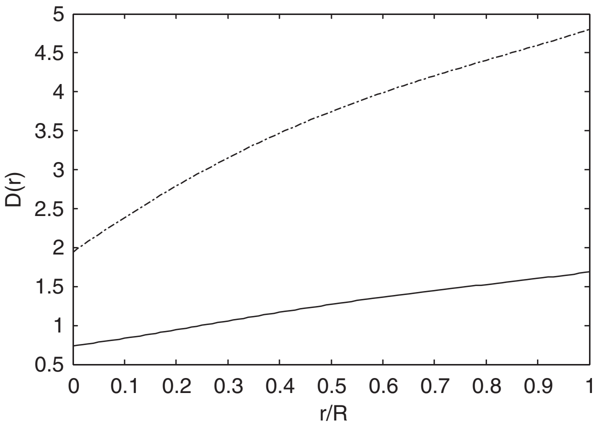

Fig. 11. Variation of $D(r)$ versus nondimensional radial coordinate r/R for $k = 7$ (-$v_r = 0.222404$, -.-$v_r = 482459$).

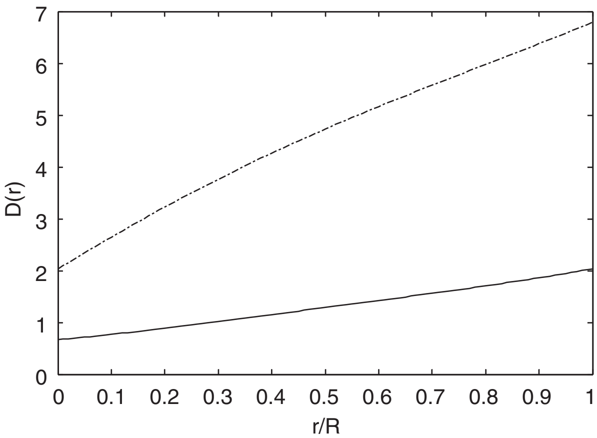

Fig. 12. Variation of $D(r)$ versus nondimensional radial coordinate r/R for $k = 8$ (-$v_r = 0.262989$, -.-$v_r = 0.578958$).

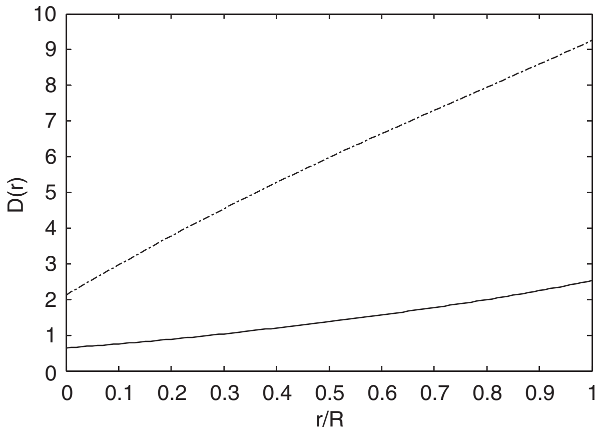

Fig. 13. Variation of $D(r)$ versus nondimensional radial coordinate r/R for $k = 9$ (-$v_r = 0.295275$, -.-$v_r = 0.656521$).

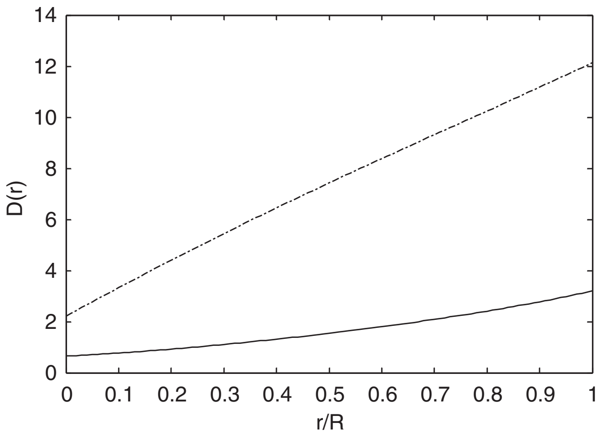

Fig. 14. Variation of $D(r)$ versus nondimensional radial coordinate r/R for $k = 10$ (-$v_r = 0.323355$, -.-$v_r = 0.717262$).

$k = 6$ corresponds to $v_r = \frac{1}{6}$ and 0.362716 (Fig. 10), the former value not corresponding to physically realizable flexural rigidity. Eq. (67) has two positive roots $v_r = 0.222404$ and 0.482459 for $k = 7$ (Fig. 11). Likewise, two roots $v_r = 0.262989$ and 0.578958 correspond to $k = 8$ (Fig. 12). The value $k = 9$ (Fig. 13) is associated with roots $v_r = 0.295275$ and 0.656521, whereas $k = 10$ (Fig. 14) is associated with $v_r = 0.323355$ and 0.717262.

7. Conclusion

Classic formula for the displacement of homogeneous polar-orthotropic circular plate, that is reported in the monograph by Lekhnitskii, is demanded to serve as an exact mode shape of the inhomogeneous plate. Closed-form solutions are derived for the corresponding natural frequencies. The paper demonstrates the rich possibilities that the inhomogeneous polar-orthotropic plates may exhibit, namely, exact coincidence of the mode shape of plate with a specified ratio k, with the static displacement of the plate with ratio equal m.

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